Duhamel's integral

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In theory of vibrations, **Duhamel's integral** is a way of calculating the response of linear systems and structures to arbitrary time-varying external excitations.

Contents

Introduction

Background

The response of a linear, viscously damped single-degree of freedom (SDF) system to a time-varying mechanical excitation $p(t)$ is given by the following second-order ordinary differential equation

$$m\frac{d^2x(t)}{dt^2}+c\frac{dx(t)}{dt}+kx(t)=p(t)$$

where m is the (equivalent) mass, x stands for the amplitude of vibration, t for time, c for the viscous damping coefficient, and k for the stiffness of the system or structure.

If a system is initially rest at its equilibrium position, from where it is acted upon by a unit-impulse at the instance t=0, i.e., $p(t)$ in the equation above is a delta function $\delta(t)$, $x(0)=\left.\frac{dx}{dt}\right|_{t=0}=0$, then by solving the differential equation one can get a fundamental solution (known as a **unit-impulse response function**)

$$h(t)=\begin{cases}\frac{1}{m\omega_d}e^{-\varsigma\omega_n t}\sin\omega_d t, & t>0\\ 0, & t<0\end{cases}$$

where $\varsigma=\frac{c}{2m\omega_n}$ is called the damping ratio of the system, ω_n is the natural circular frequency of the undamped system (when c=0) and $\omega_d=\omega_n\sqrt{1-\varsigma^2}$ is the circular frequency when damping effect is taken into account (when $c\neq 0$). If the impulse happens at $t=\tau$ instead of t=0, i.e. $p(t)=\delta(t-\tau)$, the impulse response is

$$h(t-\tau) = \frac{1}{m\omega_d} e^{-\varsigma\omega_n(t-\tau)} \sin[\omega_d(t-\tau)] \,\square\, t \geq \tau$$

Conclusion

Regarding the arbitrarily varying excitation *p*(*t*) as a superposition of a series of impulses:

then it is known from the linearity of system that the overall response can also be broken down into the superposition of a series of impulse-responses:

$$x(t) \approx \sum p(\tau) \cdot \Delta\tau \cdot h(t-\tau)$$

Letting $\Delta\tau \longrightarrow 0$, and replacing the summation by integration, the above equation is strictly valid

$$x(t) = \int_0^t p(\tau) h(t-\tau) d\tau$$

Substituting the expression of *h*(*t*-*τ*) into the above equation leads to the general expression of Duhamel's integral

$$x(t) = \frac{1}{m\omega_d} \int_0^t p(\tau) e^{-\varsigma\omega_n(t-\tau)} \sin[\omega_d(t-\tau)] d\tau$$

See also

- Duhamel's principle

References

- Ni Zhenhua, *Mechanics of Vibrations*, Xi'an Jiaotong University Press, Xi'an, 1990 (in Chinese)
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External links

- Duhamel's formula (http://tosio.math.toronto.edu/wiki/index.php/Duhamel's_formula) at "Dispersive Wiki".